

Chapter 1 - Linear Programming

Exercise 1.1

The standard form of an LP is

$$\begin{aligned} & \text{maximise} && f(\vec{x}) \\ & \text{subject to} && g_i(\vec{x}) = b_i \quad \forall i=1, \dots, m \quad (*) \\ & && h_j(\vec{x}) \leq c_j \quad \forall j=1, \dots, n \end{aligned}$$

Consider $\tilde{f}: \mathbb{R}^k \rightarrow \mathbb{R}$, $\tilde{f}(\vec{x}) = -f(\vec{x})$. Then, if the LP in (*) is feasible, i.e. $F \neq \emptyset$, $\exists \vec{x}^* \in F$ such that

$$\forall \vec{x} \in F: f(\vec{x}) \leq f(\vec{x}^*),$$

which is equivalent to

$$\forall \vec{x} \in F: \tilde{f}(\vec{x}^*) \leq \tilde{f}(\vec{x}).$$

Thus,

$$\begin{aligned} & \text{minimise} && \tilde{f}(\vec{x}) \\ & \text{subject to} && g_i(\vec{x}) = b_i \quad \forall i=1, \dots, m \quad (**) \\ & && h_j(\vec{x}) \leq c_j \quad \forall j=1, \dots, n \end{aligned}$$

has an optimal solution. Clearly, every optimal

solution of $(*)$ is an optimal solution of $(**)$ and vice versa. Also, the LP in $(*)$ is infeasible $\Leftrightarrow$ the minimisation problem in $(**)$ is infeasible.

Conclusion: an LP can either involve the maximisation or minimisation of a linear function.

Exercise 1.2

Consider the following optimization problem:

$$\begin{aligned} & \text{maximise} && f(\vec{x}) \\ & \text{subject to} && g_i(\vec{x}) = b_i \quad \forall i = 1, \dots, m \\ & && h_j(\vec{x}) \leq c_j \quad \forall j = 1, \dots, n \\ & && l_k(\vec{x}) \geq d_k \quad \forall k = 1, \dots, p \end{aligned} \quad (*)$$

where $l_k: \mathbb{R}^n \rightarrow \mathbb{R}$ is linear for each k . Define $\tilde{l}_k: \mathbb{R}^n \rightarrow \mathbb{R}$ as $-l_k$. Then each lower bound constraint implies that $\forall \vec{x} \in \mathbb{R}^n$:

$$l_k(\vec{x}) \geq d_k \Leftrightarrow \tilde{l}_k(\vec{x}) \leq -d_k.$$

Thus the problem in $(*)$ has the same feasible set as an LP with the same objective function, the same eq. constraints, the same ineq. constraints due to each h_j , and ineq. constraints due to each $\tilde{l}_k$.

Exercise 1.3

First, let us show that any linear function $f: \mathbb{R}^k \rightarrow \mathbb{R}$ can be written as $\vec{a} \cdot \vec{x}$ for some $\vec{a} \in \mathbb{R}^k$.

Linearity means that $\forall \vec{x}, \vec{y} \in \mathbb{R}^k$ and $\forall \alpha \in \mathbb{R}$:

$$f(\alpha \vec{x} + \vec{y}) = \alpha f(\vec{x}) + f(\vec{y}).$$

If $(\vec{v}_1, \dots, \vec{v}_k)$ is any basis in $\mathbb{R}^k$, we know that f is fully specified by the values $f(\vec{v}_i)$, $i=1, \dots, k$. So, set $\vec{a} \in \mathbb{R}^k$ as $(f(\vec{v}_1), \dots, f(\vec{v}_k))$. For an arbitrary $\vec{x} \in \mathbb{R}^k$:

$$\begin{aligned} \vec{a} \cdot \vec{x} &= f(\vec{v}_1)x_1 + \dots + f(\vec{v}_k)x_k \\ &= f(x_1\vec{v}_1) + \dots + f(x_k\vec{v}_k) && (f \text{ is linear}) \\ &= f\left(\sum_{i=1}^k x_i \vec{v}_i\right) && (f \text{ is linear}) \\ &= f(\vec{x}) && (\{\vec{v}_i\}_{i=1}^k \text{ is a basis}) \end{aligned}$$

Picking $(\vec{v}_1, \dots, \vec{v}_k)$ as the canonical basis in $\mathbb{R}^k$ fulfills the exercise's request.

Exercise 1.4

Take $f(x_1, x_2) = 2x_1 - x_2$. We have $f(1, 0) = 2$ and $f(0, 1) = -1$.

Consequently $\vec{a} = (2, -1)$.

Exercise 1.5

Since $g: \mathbb{R}^k \rightarrow \mathbb{R}$ is linear, isn't $(\vec{r}_i)_m = g(\vec{e}_m)$? Is that is what the exercise is asking for, it seems pointless...

Exercise 1.6

Recall that the l_1 norm of $\vec{y} \in \mathbb{R}^k$ is

$$\|\vec{y}\|_1 = \sum_{i=1}^k |y_i|.$$

It can be cast as the solution of

$$\begin{aligned} & \text{maximise } \vec{f} \cdot \vec{y} \\ & \text{subject to } \|\vec{f}\|_\infty \leq 1 \end{aligned} \quad (*)$$

The l_∞ norm of $\vec{f} \in \mathbb{R}^k$ is the optimal value of

$$\begin{aligned} & \text{minimise } \tau \\ & \text{subject to} \end{aligned} \quad (**)$$

$$-\tau \leq f_i \leq \tau \quad \forall i = 1, \dots, k.$$

We can modify (**) to check whether $\|\vec{f}\|_\infty \leq 1$:

$$\begin{aligned} & \text{minimise } \tau \\ & \text{subject to} \\ & -\tau \leq f_i \leq \tau \quad \forall i = 1, \dots, k. \\ & \tau \leq 1 \end{aligned} \quad (***)$$

minimisation of $-n \leq f_i \leq n$ forces n to be $\|f\|_\infty$. If the b_0 is greater than 1, the new constraint cannot be satisfied, making the LP infeasible. By contraposition, if it is feasible, then $\|f\|_\infty \leq 1$, which means that $-1 \leq f_i \leq 1$.

The problem in (*) has as feasible set the $\vec{f}$'s for which the LP (***) is feasible. By the previous paragraph, that means $-1 \leq f_i \leq 1$. Thus, (*) is equivalent to the LP

$$\begin{aligned} & \text{maximise } \vec{f} \cdot \vec{y} \\ & \text{subject to } -1 \leq f_i \leq 1 \quad \forall i. \end{aligned}$$

Exercise 1.7

Let $\vec{x}_1^*$ and $\vec{x}_2^*$ be optimal solutions of an LP. Choose an arbitrary $\lambda \in [0, 1]$ and set $\vec{y} = \lambda \vec{x}_1^* + (1-\lambda) \vec{x}_2^*$. By linearity of f :

$$\begin{aligned} f(\vec{y}) &= f(\lambda \vec{x}_1^* + (1-\lambda) \vec{x}_2^*) = \lambda f(\vec{x}_1^*) + (1-\lambda) f(\vec{x}_2^*) \\ &= \lambda \alpha + (1-\lambda) \alpha = \alpha. \end{aligned}$$

Thus, $\vec{y}$ is also an optimal solution.

Exercise 1.8

Check for feasibility

$$\begin{aligned} \text{find } & \vec{x} \\ \text{subject to } & x_1 + x_2 \geq 1 \\ & x_1 \leq \frac{1}{3} \\ & x_2 \leq \frac{1}{2} \end{aligned}$$

Well, if $x_1 \leq \frac{1}{3}$ and $x_2 \leq \frac{1}{2}$, then $x_1 + x_2 \leq \frac{5}{6} < 1$. Consequently, this LP is infeasible.

$$\begin{aligned} \text{find } & \vec{x} \\ \text{subject to } & x_1 + x_2 + x_3 = 1 \\ & x_1 + x_2 \geq 1 \\ & x_2 + x_3 \geq 1 \end{aligned}$$

Suppose that $x_1 + x_2 + x_3 = 1$. Then the last two constraints, if satisfied, imply $1 \geq 1 + x_3$ and $1 \geq 1 + x_1$. Thus both x_1 and x_3 are negative.

Pick $x_2 = \frac{3}{2}$, $x_1 = x_3 = -\frac{1}{4}$. Then $-\frac{1}{4} + \frac{3}{2} - \frac{1}{4} = 1$, $-\frac{1}{4} + \frac{3}{2} = \frac{5}{4} > 1$.

Exercise 1.9

(a) The feasibility problem is

$$\begin{aligned} \text{find } & P_{xyz}(x, y, z) \\ \text{subject to } & \sum_z P_{xyz}(x, y, z) = P_{xy}(x, y) \quad \forall x \forall y \end{aligned}$$

$$\sum_y P_{xyz}(x, y, z) = P_{xz}(x, z) \quad \forall x \forall z$$

$$\sum_x P_{xyz}(x, y, z) = P_{yz}(y, z) \quad \forall y \forall z$$

$$P_{xyz}(x, y, z) \geq 0 \quad \forall x \forall y \forall z$$

(b) To prove: inconsistent marginals $\Rightarrow$ infeasibility.

We prove by contradiction. Without loss of generality, assume there is a value x_0 such that

$$P_x^y(x_0) \equiv \sum_y P_{xy}(x_0, y)$$

and

$$P_x^z(x_0) \equiv \sum_z P_{xz}(x_0, z)$$

are different, and the feasibility LP has a solution $P_{xyz}(x, y, z)$. From this solution, we obtain the single-party marginal

$$P_x(x) = \sum_{y,z} P_{xyz}(x, y, z) \quad \forall x.$$

Notice that for any x :

$$P_x(x) = \sum_y P_{xy}(x, y) = \sum_z P_{xz}(x, z).$$

Consequently, for x_0 :

$$P_x(x_0) = P_x^y(x_0) = P_x^z(x_0),$$

a contradiction with the inconsistency hypothesis.

Hence, inconsistency and feasibility is a false statement. Thus, inconsistency $\Rightarrow$ infeasibility is true.

(c) We are given the marginals $P_{xy}(x,y)$ and $P_{yz}(y,z)$, which are assumed to be consistent, i.e.

$$\sum_x P_{xy}(x,y) = \sum_z P_{yz}(y,z) \quad \forall y.$$

We must construct a valid joint probability $P_{xyz}(x,y,z)$ that is consistent with the given marginals.

Since $P_{xy}(x,y)$ and $P_{yz}(y,z)$ are consistent, set

$$P_y(y) = \sum_x P_{xy}(x,y) = \sum_z P_{yz}(y,z) \quad \forall y.$$

Now, define

$$\forall x,y,z: \quad P_{xyz}(x,y,z) = \frac{P_{xy}(x,y) P_{yz}(y,z)}{P_y(y)}.$$

Since all terms on the RHS are positive, $P_{xyz}(x,y,z) \geq 0$ for every x,y,z .

Summing over x results in

$$\sum_x P_{xyz}(x,y,z) = \frac{(\sum_x P_{xy}(x,y)) P_{yz}(y,z)}{P_y(y)} = \frac{P_y(y) P_{yz}(y,z)}{P_y(y)} = P_{yz}(y,z).$$

Similarly, summing over z yields $P_{xy}(x,y)$.

Summing over x, y and z obviously reform 1, thus, due to non-negativity of $P_{xyz}(x, y, z)$, we can conclude that

$$0 \leq P_{xyz}(x, y, z) \leq 1 \quad \forall x \forall y \forall z.$$

Hence, P_{xyz} is a valid distribution that solves the given feasibility problem.

Exercise 1.10

The feasibility problem

$$\begin{aligned} &\text{find } D \\ &\text{subject to } D\vec{p} = \vec{q} \\ &\sum_x D_{xy} = 1 \quad \forall y \\ &\sum_y D_{xy} = 1 \quad \forall x \\ &D_{xy} \geq 0 \end{aligned}$$

Can be relaxed to a maximisation problem

$$\begin{aligned} &\text{maximise } \pi \\ &\text{subject to } D\vec{p} = \pi\vec{q} + (1-\pi)\vec{v} \\ &\sum_x D_{xy} = 1 \quad \forall y \\ &\sum_y D_{xy} = 1 \quad \forall x \end{aligned}$$

$$D_{xy} \geq 0$$

$$\tau \leq 1$$

Let's show that $\tau \geq 0$ does not need to be included.

Set $\tau = 0$, all we gotta do is construct the appropriate D .

For every $i \in \{1, \dots, m\}$, we have

$$\sum_{j=1}^m D_{ij} p(x_j) = \frac{1}{m}.$$

For every j , set $D_{ij} = \frac{1}{m}$. Then, the left hand side is equal to $\frac{1}{m} \sum_j p(x_j)$. Since $\vec{p}$ is normalized, the sum $\sum_j p(x_j)$ is 1.

Clearly,

$$\sum_{j=1}^m D_{ij} = \sum_{j=1}^m \frac{1}{m} = 1 \quad \forall i;$$

and

$$\sum_{i=1}^m D_{ij} = \sum_{i=1}^m \frac{1}{m} = 1 \quad \forall j.$$

Also, $D_{ij} \geq 0 \quad \forall i, \forall j$.

So, for $\tau = 0$, the feasibility problem has a solution.

Consequently, $\tau = 0$ is in the feasible set of the relaxed LP. Since this LP is a maximization problem, any optimal τ^* must then satisfy $0 \leq \tau^*$, which means that the constraint $\tau \geq 0$ is automatically satisfied by any (optimal) solution. Hence, we don't have to add this constraint to the LP.

Exercise 1.11

Let us relax the following feasibility LPs:

$$\begin{aligned} \textcircled{I} \quad & \text{find } \vec{x} \\ & \text{subject to } x_1 + x_2 \geq 1 \\ & \quad x_1 \leq \frac{1}{3} \\ & \quad x_2 \leq \frac{1}{2} \end{aligned}$$

↓

$$\begin{aligned} & \text{minimise } \tau \\ & \text{subject to } x_1 + x_2 \geq 1 - \tau \\ & \quad x_1 \leq \frac{1}{3} \\ & \quad x_2 \leq \frac{1}{2} \\ & \quad \tau \geq 0 \end{aligned}$$

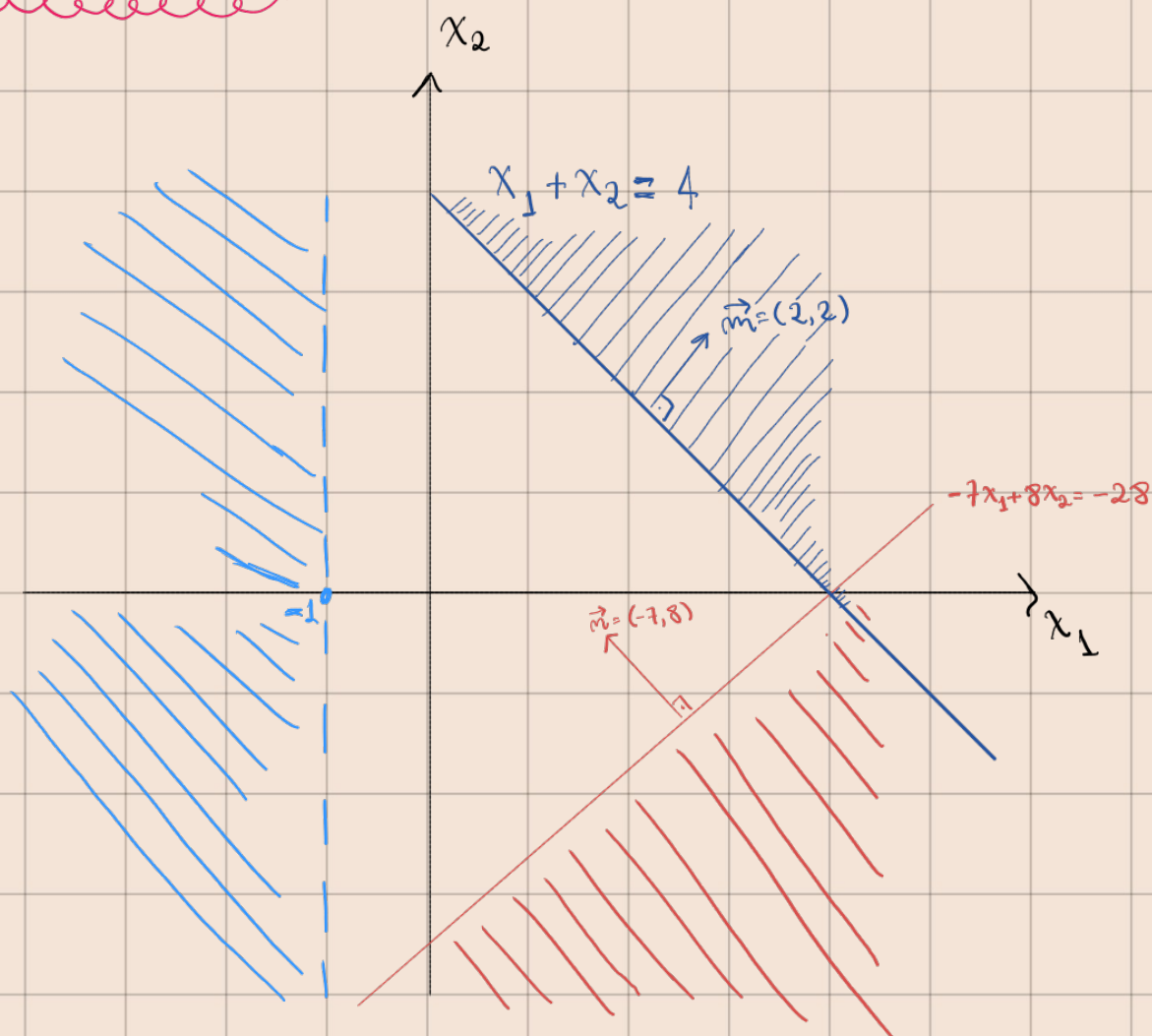
$$\begin{aligned} \textcircled{II} \quad & \text{find } \vec{x} \\ & \text{subject to } x_1 + x_2 + x_3 = 1 \\ & \quad x_1 + x_2 \geq 1 \\ & \quad x_2 + x_3 \geq 1 \end{aligned}$$

↓

$$\begin{aligned} & \text{minimise } \tau \\ & \text{subject to } x_1 + x_2 + x_3 = 1 \\ & \quad x_1 + x_2 \geq 1 - \tau \\ & \quad x_2 + x_3 \geq 1 - \tau \\ & \quad \tau \geq 0 \end{aligned}$$

In each LP, if $\tau^* = 0$, then the original feasibility problem has a solution. I solved both using Cvxpy + mumpy. For problem 1, $\tau^* = \frac{1}{6}$, so the original problem is infeasible. For problem 2, $\tau^* = 0$.

Exercise 1.12



Constraints:

$$x_1 \leq -1$$

$$-7x_1 + 8x_2 \leq -28$$

$$x_1 + x_2 \geq 4$$

If $x_1 \leq -1$, then the 2nd constraint implies that $x_2 \leq 0$, whereas the 3rd one implies that $x_2 \geq 0$. So, minimisation or maximisation of any linear function $f(x_1, x_2)$ have no optimal solutions, since the feasible set is empty.

Exercise 1.13

Consider the following LP:

$$\begin{aligned} &\text{minimise } x_1 \\ &\text{subject to } -7x_1 + 8x_2 \leq 28 \\ &\quad \quad \quad x_1 + x_2 \geq 4 \end{aligned}$$

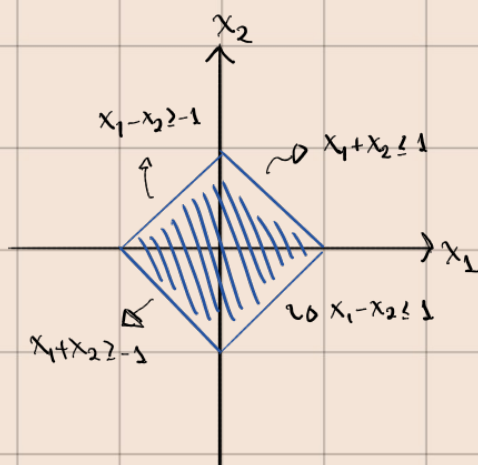
This LP is feasible and $\vec{x}^* = (4, 0)$. It becomes infeasible if we add a constraint such as $-x_1 + x_2 = 2$

Exercise 1.14

(a) LP:

$$\begin{aligned} &\text{maximise } 2x_1 + x_2 \\ &\text{subject to } x_1 + x_2 \leq 1 \\ &\quad \quad \quad x_1 - x_2 \leq 1 \\ &\quad \quad \quad x_1 + x_2 \geq -1 \\ &\quad \quad \quad x_1 - x_2 \geq -1 \end{aligned}$$

Feasible set:

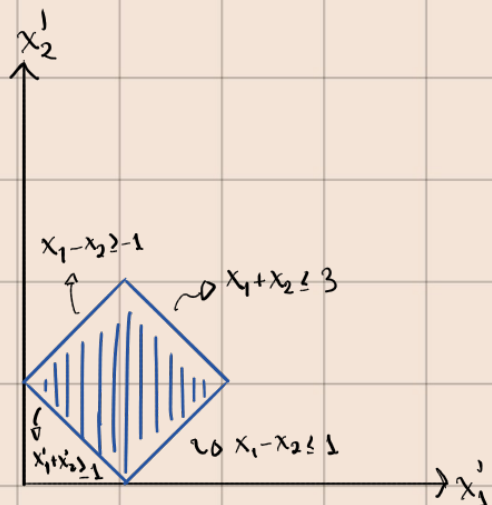


Clearly, any point $\vec{x} \in F$ outside of the 1st quadrant doesn't satisfy $\vec{x} \geq 0$!

(b) Define $\vec{x}' = \vec{x} + \vec{1}$. In terms of this shifted variable, the LP becomes

$$\begin{aligned} &\text{maximise} && 2x'_1 + x'_2 \\ &\text{subject to} && x'_1 + x'_2 \leq 3 \\ &&& x'_1 - x'_2 \leq 1 \\ &&& x'_1 + x'_2 \geq 1 \\ &&& x'_1 - x'_2 \geq -1 \end{aligned}$$

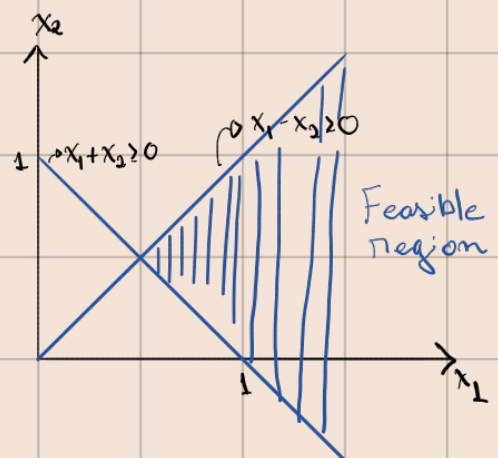
The feasible set F' is



Now every $\vec{x}' \in F'$ satisfies $\vec{x}' \geq 0$.

(c) LP:

$$\begin{aligned} &\text{maximise} && -3x_1 + x_2 \\ &\text{subject to} && x_1 + x_2 \geq 0 \\ &&& x_1 - x_2 \geq 0 \end{aligned}$$



(d) Set $x'_1 = \frac{x_1 - x_2}{\sqrt{2}}$ and $x'_2 = \frac{x_1 + x_2}{\sqrt{2}}$. Then: $x_1 = \frac{x'_1 + x'_2}{\sqrt{2}}$ and $x_2 = \frac{x'_2 - x'_1}{\sqrt{2}}$.

Consequently:

$$\bullet -3x_1 + x_2 = \frac{-3x'_1 - 3x'_2 + x'_2 - x'_1}{\sqrt{2}} = \frac{-4x'_1 - 2x'_2}{\sqrt{2}} = \sqrt{2}(2x'_1 - x'_2)$$

$$\bullet x_1 + x_2 = \sqrt{2} x'_2$$

$$\bullet x_1 - x_2 = \sqrt{2} x'_1$$

Using these, we can rewrite the LP as

$$\text{maximise } 2x'_1 - x'_2$$

$$\text{subject to } x'_1 \geq 0$$

$$x'_2 \geq 0$$

The change of coordinates is given by a $\pi/4$ rotation:

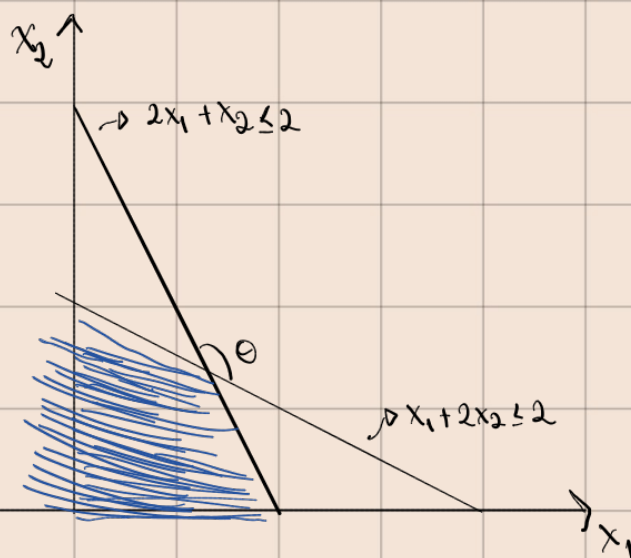
$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}}_M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Clearly, $\det M = \cos^2(\pi/4) + \sin^2(\pi/4) = 1$. Also:

$$\boxed{\vec{y}' \cdot \vec{x}' = \vec{y}'^T (M^T M \vec{x}) = \vec{y}'^T \vec{x} = \vec{y} \cdot \vec{x}}$$

(e) LP: Maximise $x_1 + x_2$
 subject to $2x_1 + x_2 \leq 2$
 $x_1 + 2x_2 \leq 2$

Feasible region



- Since the feasible region is unbounded, no shift can map all $\vec{x} \in F$ to $\vec{x} \geq 0$.
- Any rotation or reflection preserves angles. Now, since the angle θ is greater than $\pi/2$, the angle between the rotated or reflected hyperplanes will still be $\theta > \pi/2$, hence at some point x_1 or x_2 will become negative.

(f) Set

$$x'_1 = -4x_1 - 2x_2 + 4, \quad x'_2 = -2x_1 - 4x_2 + 4$$

In matrix form:

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Consequently:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1' - 4 \\ x_2' - 4 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-4x_1' + 2x_2'}{12} \\ \frac{2x_1' - 4x_2'}{12} \end{pmatrix} + \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$$

That is: $x_1 = \frac{1}{12}(-4x_1' + 2x_2' + 8)$ and $x_2 = \frac{1}{12}(2x_1' - 4x_2' + 8)$

* Objective function: $x_1 + x_2 = \frac{1}{12}(-2x_1' - 2x_2' + 16) = \boxed{-\frac{1}{6}(x_1' + x_2')} + \frac{4}{3}$

* Constraints: $2x_1 + x_2 \leq 2 \Rightarrow \frac{1}{12}(-8x_1' + 4x_2' + 16 + 2x_1' - 4x_2' + 8) \leq 2 \equiv \boxed{6x_1' \geq 0}$

$$x_1 + 2x_2 \leq 2 \Rightarrow \frac{1}{12}(-4x_1' + 2x_2' + 4x_1' - 8x_2' + 24) \leq 2 \equiv \boxed{6x_2' \geq 0}$$

(g) $S = \begin{pmatrix} -4 & 2 \\ -2 & -4 \end{pmatrix}$, $\det S = 12 \neq 0 \Rightarrow \exists S^{-1}$ and $S \notin O(2)$

$$S^{-1} = \frac{1}{12} \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}, \vec{x} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Exercise 1.15

$$\begin{array}{ll|ll}
 \text{Primal:} & \text{maximise} & \vec{a} \cdot \vec{x} & \text{Dual:} & \text{minimise} & \vec{b} \cdot \vec{y} + \vec{c} \cdot \vec{z} \\
 & \text{subject to} & B\vec{x} = \vec{b} & & \text{subject to} & \vec{z} \geq 0 \\
 & & C\vec{x} \leq \vec{c} & & & B^T \vec{y} + C^T \vec{z} = \vec{a}
 \end{array}$$

We could simplify the dual LP further by setting $\vec{y}' = \vec{y} \oplus \vec{z} \in \mathbb{R}^m \oplus \mathbb{R}^m \simeq \mathbb{R}^{m+m}$.

Write $\vec{b}' = \vec{b} \oplus \vec{c}$, $C' = \mathbf{0}_m \oplus \mathbf{I}_m$, $A = \underbrace{(B^T \ C^T)}_{\text{Concatenation}}$. The dual now reads as

$$\begin{array}{ll}
 \text{minimise} & \vec{b}' \cdot \vec{y}' \\
 \text{subject to} & A\vec{y}' = \vec{a} \\
 & C'\vec{y}' \geq 0
 \end{array}$$

Exercise 1.16

$$\begin{array}{ll}
 \text{Primal:} & \text{maximise} \quad \tau \\
 & \text{subject to} \quad D\vec{p} = \tau \vec{q} + (1-\tau)\vec{v} \quad (\vec{p}, \vec{q}, \vec{v} \in \mathbb{R}^n) \\
 & \sum_x D_{xy} = 1 \quad \forall y \\
 & \sum_y D_{xy} = 1 \quad \forall x \\
 & D_{xy} \geq 0 \quad \forall x \forall y \\
 & \tau \leq 1
 \end{array}$$

$$(\tau, D_{11}, \dots, D_{1m}, \dots, D_{m1}, \dots, D_{mm}) \equiv \vec{v}$$

$$\vec{I}_m^{(1)} = 0 \oplus \vec{I}_m \oplus \vec{0}_m \oplus \dots \oplus \vec{0}_m, \dots, \vec{I}_m^{(m)} = 0 \oplus \vec{0}_m \oplus \dots \oplus \vec{I}_m$$

$$0) \vec{a} = 1 \oplus \vec{0}_m^{(1)} \oplus \vec{0}_m^{(2)} \oplus \dots \oplus \vec{0}_m^{(m)} \rightarrow \tau = \vec{a} \cdot \vec{v}$$

$$1) \sum_y D_{xy} p_y = q_x + (1-q) u_x = \vec{\pi}_x \cdot \vec{v} = u_x$$

$$(u_x - q_x) \tau + \sum_y p_y D_{xy} = u_x$$

$$\bullet \vec{\pi}_x = (u_x - q_x) \oplus \vec{0}_m^{(1)} \oplus \dots \oplus \vec{0}_m^{(x-1)} \oplus \vec{p} \oplus \vec{0}_m^{(x+1)} \oplus \dots \oplus \vec{0}_m^{(m)}$$

$$2) \sum_x D_{xy} = 1 \quad \forall y \equiv (0 \oplus \vec{e}_y^{(1)} \oplus \vec{e}_y^{(2)} \oplus \dots \oplus \vec{e}_y^{(m)}) \cdot \vec{v} \equiv \vec{E}_y \cdot \vec{v} = 1 \quad \forall y$$

$$3) \sum_y D_{xy} = 1 \quad \forall x \equiv \vec{I}_m^{(x)} \cdot \vec{v} = 1 \quad \forall x$$

$$4) D_{xy} \geq 0 \quad \forall x \forall y \equiv -\vec{e}_y^{(x)} \cdot \vec{v} \leq 0 \quad \forall x \forall y$$

$$5) \tau \leq 1 \equiv \vec{s} \cdot \vec{v} \leq 1, \vec{s} = \vec{a}$$

The matrix B is given by:

$$B = \begin{pmatrix} \vec{\pi}_x^T \\ \vec{E}_y^T \\ \vec{I}_m^{(x)T} \end{pmatrix} \in M_{3m \times (m^2+1)}$$

The matrix C is given by

$$C = \begin{pmatrix} \vec{s}^T \\ -\vec{e}_y^{(x)T} \end{pmatrix} \in M_{(m^2+1) \times (m^2+1)}$$

$$\text{We have } \vec{b} = \begin{pmatrix} \vec{v} \\ \vec{I}_m \\ \vec{I}_m \end{pmatrix}, \vec{c} = \begin{pmatrix} 1 \\ \vec{0}_{2m} \end{pmatrix}$$

Thus, the dual LP is

$$\text{minimise } \overrightarrow{b \cdot \vec{\gamma}} = \frac{1}{n} \sum_{i=1}^m \gamma_i + \sum_{i=m+1}^{3n} \gamma_i + z_1$$

$$\text{subject to } \vec{z} \geq 0$$

$$\underbrace{(\vec{p}_1 \dots \vec{p}_m \vec{e}_1 \dots \vec{e}_m \vec{1}_m^{(1)} \dots \vec{1}_m^{(m)})}_{\begin{pmatrix} u_1 - q_1 & u_m - q_m & 0 & \dots & 0 & 0 & \dots & 0 \\ \vec{p}_1 & \vec{0}_m & \vec{e}_1 & \dots & \vec{e}_m & \vec{1}_m & \dots & \vec{0}_m \\ \vec{0}_m & \vec{0}_m & \vec{e}_1 & \dots & \vec{e}_m & \vec{0}_m & \dots & \vec{0}_m \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vec{0}_m & \vec{p}_1 & \vec{e}_1 & \dots & \vec{e}_m & \vec{0}_m & \dots & \vec{1}_m \end{pmatrix}} \vec{\gamma} + \underbrace{(\vec{1} - \vec{e}_1^{(1)} \dots \vec{e}_m^{(1)} \dots \vec{e}_1^{(m)} \dots \vec{e}_m^{(m)})}_{\begin{pmatrix} 1 & & & & & & & \\ & -\mathbb{I}_m & & & & & & \\ & & \ddots & & & & & \\ & & & -\mathbb{I}_m & & & & \end{pmatrix}} \vec{z} = \vec{1} \oplus \vec{0}_{m^2}$$

So, the $m^2 + 1$ constraints are:

- $\sum_{i=1}^m (u_i - q_i) \gamma_i + z_1 = 1 \leadsto$ equivalent to $\frac{1}{n} \sum_{i=1}^m \gamma_i - \sum_{i=1}^m q_i \gamma_i + z_1 = 1$
- $P_1 \gamma_1 + \gamma_{m+1} + \gamma_{2m+1} - z_2 = 0$
- $P_2 \gamma_1 + \gamma_{m+2} + \gamma_{2m+2} - z_3 = 0$
- $\vdots$
- $P_m \gamma_1 + \gamma_{2m} + \gamma_{2m+1} - z_{m+1} = 0$
- $P_1 \gamma_2 + \gamma_{m+1} + \gamma_{2m+2} - z_{m+2} = 0$
- $\vdots$
- $P_m \gamma_2 + \gamma_{2m} + \gamma_{2m+2} - z_{2m+1} = 0$
- $\vdots$
- $P_1 \gamma_m + \gamma_{m+1} + \gamma_{2m} - z_{m(m-1)} = 0$
- $P_m \gamma_m + \gamma_{2m} + \gamma_{2m} - z_{m^2+1} = 0$

Plus the $m^2 + 1$ positivity constraints $z_i \geq 0$.

For $i \geq 2$, z_i is a slack variable; we can solve each of them in terms of components of $\vec{\gamma}$ and obtain

inequality constraints:

- $P_1 \gamma_1 + \gamma_{m+1} + \gamma_{2m+1} \geq 0$
- $P_2 \gamma_1 + \gamma_{m+2} + \gamma_{2m+2} \geq 0$
- $\vdots$
- $P_m \gamma_1 + \gamma_{2m} + \gamma_{2m+1} \geq 0$
- $P_1 \gamma_2 + \gamma_{m+1} + \gamma_{2m+2} \geq 0$
- $\vdots$
- $P_m \gamma_2 + \gamma_{2m} + \gamma_{2m+2} \geq 0$
- $\vdots$
- $P_1 \gamma_m + \gamma_{m+1} + \gamma_{2m} \geq 0$
- $P_m \gamma_m + \gamma_{2m} + \gamma_{2m} \geq 0$

Dual LP:

$$\text{minimise } 1 + \sum_{i=1}^m g_i \gamma_i + \sum_{i=m+1}^{2m} \gamma_i$$

$$\text{subject to } \sum_{i=1}^m \left(\frac{1}{m} - g_i \right) \gamma_i \leq 1$$

$$P_i \gamma_j + \gamma_{m+i} + \gamma_{2m+j} \geq 0 \quad \forall i, j = 1, \dots, m$$

Exercise 1.17

Primal: minimise $\vec{a} \cdot \vec{x}$

subject to $B \vec{x} = \vec{b}$

$C \vec{x} \geq \vec{c}$

$$B = \begin{pmatrix} \vec{p}_1^T \\ \vdots \\ \vec{p}_m^T \end{pmatrix}, C = \begin{pmatrix} \vec{s}_1^T \\ \vdots \\ \vec{s}_m^T \end{pmatrix}$$

Lagrangian:

$$L(\vec{x}, \vec{\gamma}, \vec{\beta}) = \vec{a} \cdot \vec{x} + \sum_{i=1}^m \gamma_i (b_i - \vec{p}_i \cdot \vec{x}) + \sum_{j=1}^m \beta_j (c_j - \vec{s}_j \cdot \vec{x}) \quad (*)$$

$$= \left(\vec{a} - \sum_{i=1}^m \gamma_i \vec{p}_i + \sum_{j=1}^m \beta_j \vec{s}_j \right) \cdot \vec{x} + \sum_{i=1}^m \gamma_i b_i + \sum_{j=1}^m \beta_j c_j \quad (**)$$

In the feasible region, $c_j - \vec{s}_j \cdot \vec{x} \leq 0$ and $b_i - \vec{p}_i \cdot \vec{x} = 0$.

Thus, if $\beta_j \geq 0$, $\vec{a} \cdot \vec{x} + \sum_{j=1}^m \beta_j (c_j - \vec{s}_j \cdot \vec{x}) \leq \vec{a} \cdot \vec{x} \quad \forall \vec{x} \in F$.

Let $g(\vec{\gamma}, \vec{\beta}) = \inf_{\vec{x} \in \mathbb{R}^d} L(\vec{x}, \vec{\gamma}, \vec{\beta})$. We have $g(\vec{\gamma}, \vec{\beta}) \leq \inf_{\vec{x} \in F} L(\vec{x}, \vec{\gamma}, \vec{\beta}) \leq \vec{a} \cdot \vec{x}^*$
 $\forall \vec{\gamma}, \vec{\beta}$.

and $g(\vec{\gamma}, \vec{\beta})$ is bounded (i.e. the inf. is finite) iff $\vec{a} = B^T \vec{\gamma} + C^T \vec{\beta}$.

Hence by maximizing $g(\vec{\gamma}, \vec{\beta})$ we get a lower bound for $\vec{a} \cdot \vec{x}^*$.

The dual LP is

$$\begin{aligned} & \text{Maximise} \quad \vec{b} \cdot \vec{\gamma} + \vec{c} \cdot \vec{\beta} \\ & \text{subject to} \quad B^T \vec{\gamma} + C^T \vec{\beta} = \vec{a} \\ & \quad \quad \quad \vec{\beta} \geq 0 \end{aligned}$$

Exercise 1.18

Suppose we have a constraint

$$\vec{s}_j \cdot \vec{x} \leq c_j.$$

Let us introduce a variable w such that

$$w + \vec{s}_j \cdot \vec{x} - c_j = 0.$$

The initial constraint becomes a positivity condition on w :

$$w = c_j - \vec{s}_j \cdot \vec{x} \geq 0.$$

Exercise 1.19

Suppose that

$$c_i - \vec{s}_i \cdot \vec{x}^* > 0 \quad (1)$$

for some $i \in \{1, \dots, m\}$. If strong duality holds, then we have

$$z_i^* (c_i - \vec{s}_i \cdot \vec{x}^*) = 0.$$

Due to (1), the above can only be true if $z_i^* = 0$.

Exercise 1.20

Primal: maximise $x_1 + 2x_2$
 subject to $-1 \leq x_1 \leq 2$
 $-3 \leq x_2 \leq 4$

(a) By inspection, $x^* = (2, 4)$, which yields $\alpha = 10$.

(b) Dual variables:

$$z_1: -1 \leq x_1, z_2: x_1 \leq 2, z_3: -3 \leq x_2, z_4: x_2 \leq 4.$$

Lagrangian:

$$L(x_1, x_2, \vec{z}) = x_1 + 2x_2 + z_1(1+x_1) + z_2(2-x_1) + z_3(3+x_2) + z_4(4-x_2)$$

$$= x_1 + (z_1 - z_2)x_1 + 2x_2 + (z_3 - z_4)x_2 + z_1 + 2z_2 + 3z_3 + 4z_4$$

$$= [1 - (z_2 - z_1)]x_1 + [2 - (z_4 - z_3)]x_2 + z_1 + 2z_2 + 3z_3 + 4z_4$$

Dual LP:

$$\text{minimise } z_1 + 2z_2 + 3z_3 + 4z_4$$

$$\text{subject to } z_2 - z_1 = 1$$

$$z_4 - z_3 = 2$$

$$z_i \geq 0, i = 1, \dots, 4$$

(c) By inspection we can take $z_1^* = z_3^* = 0, z_2^* = 1, z_4^* = 2$.

Thus $\beta = 10$ (strong duality holds).

Complementary slackness:

$$\bullet z_1^*(1+x_1^*) = 0 \text{ with } z_1^* = 0 \text{ and } x_1^* = 2$$

$$\bullet z_2^*(2-x_1^*) = 0 \text{ with } z_2^* = 1 \text{ and } x_1^* = 2$$

- $z_3^*(3+x_2^*)=0$ with $z_3^*=0$ and $x_2^*=4$
- $z_4^*(4-x_2^*)=0$ with $z_4^*=2$ and $x_2^*=4$

Active constraints:	Primal	Dual
	$x_1 \leq 2$	$z_1 \geq 0$
	$x_2 \leq 4$	$z_3 \geq 0$

Inactive constraints:	Primal	Dual
	$-1 \leq x_1$	$z_2 \geq 0$
	$-3 \leq x_2$	$z_4 \geq 0$

(d) Primal LP without inactive constraints:

$$\begin{aligned}
 &\text{maximise } x_1 + 2x_2 \\
 &\text{subject to } x_1 \leq 2 \\
 &\quad \quad \quad x_2 \leq 4
 \end{aligned}$$

Lagrangian:

$$\begin{aligned}
 L &= x_1 + 2x_2 + z_1(2-x_1) + z_2(4-x_2) \\
 &= (1-z_1)x_1 + (2-z_2)x_2 + 2z_1 + 4z_2
 \end{aligned}$$

$$\begin{aligned}
 \text{Dual:} \quad &\text{minimise } 2z_1 + 4z_2 \\
 &\text{subject to } (z_1, z_2) = (1, 2) \\
 &\quad \quad \quad z_1 \geq 0, z_2 \geq 0
 \end{aligned}$$

Exercise 1.21

Lagrangian: $\vec{1} \cdot \vec{x} - \vec{u} \cdot (\vec{x} + \vec{y}) - \vec{v} \cdot (\vec{x} - \vec{y}) = (\vec{1} - \vec{u} - \vec{v}) \cdot \vec{x} + (\vec{v} - \vec{u}) \cdot \vec{y}$

(a) For feasible $\vec{x}$: $\vec{x} + \vec{y} \geq 0$ and $\vec{x} - \vec{y} \geq 0$. For feasible $\vec{u}, \vec{v}$: $\vec{u} \geq 0$ and $\vec{v} \geq 0$. Since optimal points are feasible, they satisfy the constraints. Thus, if strong duality holds:

$$\vec{u}^* \cdot (\vec{x}^* + \vec{y}) = 0 \quad \text{and} \quad \vec{v}^* \cdot (\vec{x}^* - \vec{y}) = 0.$$

Since all terms have non-negative components, we obtain

$$\boxed{u_i^*(x_i^* + y_i) = 0, \quad v_i^*(x_i^* - y_i) = 0 \quad \forall i}$$

↳ Comp. slackness conditions

(b) Pick $x_i = |y_i|$. Since $y_i \in \mathbb{R}$, we have

$$\left\{ \begin{array}{l} y_i = |y_i| \text{ if } y_i \geq 0 \text{ or } y_i = -|y_i| \text{ if } y_i < 0. \end{array} \right.$$

$$\text{So, if } y_i \geq 0: -|y_i| \leq |y_i| = y_i \approx -|y_i| \leq y_i$$

$$\text{if } y_i < 0: y_i = -|y_i| < 0 < |y_i|$$

Thus $-|y_i| \leq y_i \leq |y_i| \quad \forall y_i \in \mathbb{R} \therefore x_i = |y_i|$ is feasible.

Pick $u_i = \begin{cases} 1 & \text{if } y_i < 0 \\ 0 & \text{otherwise} \end{cases}$. Clearly, $u_i \geq 0 \forall i$.

Pick $v_i = \begin{cases} 1 & \text{if } y_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$. Clearly, $v_i \geq 0 \forall i$.

Now, $u_i + v_i$ is always equal to 1. If $y_i < 0$, then $u_i = 1$ and $v_i = 0$, whereas $y_i \geq 0$ implies that $u_i = 0$ and $v_i = 1$. Thus: $\vec{u} + \vec{v} = \vec{1}$. This choice of $\vec{u}$ and $\vec{v}$ is feasible.

(C) If $\vec{x}$ is feasible, we have

$$-x_i \leq y_i \leq x_i \quad \forall i.$$

If $x_i \geq 0$, this is equivalent to

$$|y_i| \leq x_i. \quad (*)$$

Now, any feasible $\vec{x}$ has non-negative components:

$$-x_i \leq y_i \leq x_i \Rightarrow 0 \leq y_i + x_i \leq 2x_i \Rightarrow 0 \leq 2x_i \Rightarrow 0 \leq x_i.$$

Thus, (*) holds for all feasible $\vec{x}$, which means that $x_i^* = |y_i|$.

$$\text{Hence, } \vec{1} \cdot \vec{x}^* = \sum_i |y_i| = \|\vec{y}\|_1.$$

Ok, actually the exercise is asking us to check that the choices from (b) satisfy complementary slackness. Hence, they are optimal. This can be done by direct computation!

Once we check this, again we have $\vec{1} \cdot \vec{x}^* = \|\vec{y}\|_1$.

Also:

$$(\vec{v}^* - \vec{v}) \cdot \vec{y} = \vec{v}^* \cdot \vec{y} - \vec{v} \cdot \vec{y} = \sum_{y_i \geq 0} y_i + \sum_{y_i < 0} (-y_i) = \sum_i |y_i| = \|\vec{y}\|_1.$$

Exercise 1.22

Primal: minimise τ
subject to $-\vec{1}\tau \leq \vec{y} \leq \vec{1}\tau$

(a)

Lagrangian:
$$L = \tau - \vec{v} \cdot (\vec{1}\tau + \vec{y}) - \vec{v} \cdot (\vec{1}\tau - \vec{y})$$

$$= [1 - (\vec{v} + \vec{v}) \cdot \vec{1}] \tau + (\vec{v} - \vec{v}) \cdot \vec{y}$$

Dual: maximise $(\vec{v} - \vec{v}) \cdot \vec{y}$
subject to $(\vec{v} + \vec{v}) \cdot \vec{1} = 1$
 $\vec{v} \geq 0, \vec{v} \geq 0$

(b) Fact: $\|\vec{y}\|_\infty = \text{maximise } \vec{1} \cdot \vec{y}$
subject to $\|\vec{1}\|_1 \leq 1$.

We can pick $\vec{x} = \vec{v} - \vec{v}$ s.t. $(\vec{v} + \vec{v}) \cdot \vec{1} = 1, \vec{v} \geq 0, \vec{v} \geq 0$.

If has $\|\vec{x}\|_1 \leq 1$:

$$\bullet \|\vec{v} - \vec{v}\|_1 = \sum_i |v_i - v_i| \leq \sum_i |v_i| + \sum_i |v_i| = \sum_i (v_i + v_i) = (\vec{v} + \vec{v}) \cdot \vec{1} = 1.$$

Thus, this choice of $\vec{x}$ always belongs to B_1 .

Now, suppose that $\vec{x} \in B_1$, i.e. $\|\vec{x}\|_1 \leq 1$. $\vec{x}$ has positive and negative components. Choose $\vec{x}^{(+)}$ according to

$$x_i^{(+)} = \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and $\vec{x}^{(-)}$ according to

$$x_i^{(-)} = \begin{cases} -x_i & \text{if } x_i < 0 \\ 0 & \text{otherwise.} \end{cases}$$

This results in $x_i^{(+)} \geq 0$, $x_i^{(-)} \geq 0$ and $\vec{x} = \vec{x}^{(+)} - \vec{x}^{(-)}$. However, $(\vec{x}^{(+)} + \vec{x}^{(-)}) \cdot \vec{1} = \|\vec{x}\|_1 \leq 1$. To fix this, set $\delta = 1 - \|\vec{x}\|_1 \geq 0$.

Then, define $\vec{\delta} = \frac{\delta}{2d} \vec{1}$ and

$$\vec{v} = \vec{x}^{(-)} + \vec{\delta}, \quad \vec{w} = \vec{x}^{(+)} + \vec{\delta}.$$

Their difference is still equal to $\vec{x}$ and they are positive.

Lastly:

$$\begin{aligned} (\vec{v} + \vec{w}) \cdot \vec{1} &= (\vec{x}^{(-)} + \vec{x}^{(+)}) \cdot \vec{1} + 2\vec{\delta} \cdot \vec{1} \\ &= \|\vec{x}\|_1 + 2 \cdot \left(\frac{\delta}{2d}\right) \vec{1} \cdot \vec{1} = \|\vec{x}\|_1 + \frac{\delta}{d} d = \|\vec{x}\|_1 + 1 - \|\vec{x}\|_1 = 1. \quad \square \end{aligned}$$

Exercise 1.23

Primal: $\|\vec{y}\|_1 = \text{maximise } \vec{p} \cdot \vec{y}$

subject to $\|\vec{p}\|_\infty \leq 1$

$\|\vec{f}\|_\infty \leq 1 \Leftrightarrow -\vec{1} \leq \vec{f} \leq \vec{1}$, thus we have 2d constraints. Let α_i and β_i be variables associated to $-1 \leq f_i$ and $f_i \leq 1$, respectively.

$$L = \vec{y} \cdot \vec{f} + \sum_{i=1}^m \alpha_i (1 + f_i) + \sum_{i=1}^m \beta_i (1 - f_i) = (\vec{y} + \vec{\alpha} - \vec{\beta}) \cdot \vec{f} + \sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i$$

$$\begin{array}{ll} \text{minimise} & (\vec{\alpha} + \vec{\beta}) \cdot \vec{1} \\ \text{subject to} & \vec{\beta} - \vec{\alpha} = \vec{y} \\ & \vec{\alpha} \geq 0, \vec{\beta} \geq 0 \end{array}$$

Exercise 1.24

Primal 1: minimise f

s.t.

$$D\vec{p} = \vec{q}'$$

$$\|\vec{q} - \vec{q}'\|_1 \leq f$$

$$\sum_y D_{xy} = 1 \quad \forall x$$

$$\sum_x D_{xy} = 1 \quad \forall y$$

$$D_{xy} \geq 0 \quad \forall x, y$$

Primal 2: minimise f

s.t.

$$D\vec{p} = \vec{q}'$$

$$\|\vec{q} - \vec{q}'\|_\infty \leq f$$

$$\sum_y D_{xy} = 1 \quad \forall x$$

$$\sum_x D_{xy} = 1 \quad \forall y$$

$$D_{xy} \geq 0 \quad \forall x, y$$

- $\|\vec{q} - \vec{q}'\|_1 \leq \tau \Leftrightarrow \frac{1}{\tau}(\vec{q} - \vec{q}') \in B_1 \Leftrightarrow \frac{1}{\tau}(\vec{q} - \vec{q}') = \vec{v} - \vec{u}, (\vec{v} + \vec{u}) \cdot \vec{1} = 1, \vec{v} \geq 0, \vec{u} \geq 0.$

$$\Rightarrow \vec{q} = \vec{q}' + \overbrace{\tau \vec{v}}^{\vec{\alpha}} - \overbrace{\tau \vec{u}}^{\vec{\beta}} = \vec{q}' + \vec{\alpha} - \vec{\beta}, (\vec{\alpha} + \vec{\beta}) \cdot \vec{1} = \tau, \vec{\alpha} \geq 0, \vec{\beta} \geq 0$$

Thus, we can rewrite the first primal LP as

minimise f

s.t. $D\vec{p} = \vec{q}'$

$$\vec{q}' + \vec{\alpha} - \vec{\beta} = \vec{q}$$

$$(\vec{\alpha} + \vec{\beta}) \cdot \vec{1} = \tau \quad \text{guarantees that } \tau \geq 0$$

$$\sum_y D_{xy} = 1 \quad \forall x$$

$$\sum_x D_{xy} = 1 \quad \forall y$$

$$D_{xy} \geq 0 \quad \forall x, \forall y$$

$$\vec{\alpha} \geq 0, \vec{\beta} \geq 0$$

- $\|\vec{q} - \vec{q}'\|_\infty \leq \tau \Leftrightarrow \frac{1}{\tau}(\vec{q} - \vec{q}') \in B_\infty \Leftrightarrow \vec{q} = \vec{q}' + \tau \vec{v} - \tau \vec{u}, \vec{v} + \vec{u} = \vec{1}, \vec{v} \geq 0, \vec{u} \geq 0$

$$\Leftrightarrow \vec{q} = \vec{q}' + \vec{\alpha} - \vec{\beta}, \vec{\alpha} + \vec{\beta} = \tau \vec{1}, \vec{\alpha} \geq 0, \vec{\beta} \geq 0$$

Thus, the second LP is:

minimise f

s. f. $D\vec{p} = \vec{q}'$

$$\vec{q}' + \vec{\alpha} - \vec{\beta} = \vec{q}$$

$$\vec{\alpha} + \vec{\beta} = \tau \vec{1}$$

$$\sum_y D_{xy} = 1 \quad \forall x$$

$$\sum_x D_{xy} = 1 \quad \forall y$$

$$D_{xy} \geq 0 \quad \forall x \forall y$$

$$\vec{\alpha} \geq 0, \vec{\beta} \geq 0$$

Chapter 2 - Semidefinite Programming

Problem 2.1

(a) Let $A, X \in \text{Her}(\mathbb{C}^d)$ for some $d \in \mathbb{N}$ and choose any orthonormal basis $(v_1, \dots, v_d) \in \mathbb{C}^d$. Then, the matrix of AX in this basis has entries

$$(AX)_{ij} = \sum_{k=1}^d A_{ik} X_{kj}$$

where $A_{ik} = \langle v_i, Av_k \rangle$ and $X_{ij} = \langle v_i, Xv_j \rangle$. The trace of AX is given by

$$\text{Tr}(AX) = \sum_{i=1}^d A_{ii} = \sum_{i=1}^d \sum_{k=1}^d A_{ik} X_{ki}.$$

Now,

$$\text{Tr}(AX)^* = \left(\sum_{i=1}^d \sum_{k=1}^d A_{ik} X_{ki} \right)^* = \sum_{i=1}^d \sum_{k=1}^d A_{ik}^* X_{ki}^*.$$

Since A and X are hermitian, we have $A_{ik}^* = A_{ki}$ and $X_{ki}^* = X_{ik}$.

Thus

$$\text{Tr}(AX)^* = \sum_{i=1}^d \sum_{k=1}^d A_{ki} X_{ik} = \sum_{i=1}^d \sum_{k=1}^d A_{ik} X_{ki} = \text{Tr}(AX).$$

Hence, $\text{Tr}(AX) \in \mathbb{R}$.

(b) Suppose $f: \text{Her}(\mathbb{C}^d) \rightarrow \mathbb{R}$ is given by $f(X) = \text{Tr}(AX)$ for some $A \in \text{Her}(\mathbb{C}^d)$. Let $(u_1, \dots, u_d)$ be an ONB in $\mathbb{C}^d$. Then

$$\begin{aligned} f(u_j^\dagger u_i) &= \text{Tr}(A u_j u_i^\dagger) \\ &= \sum_{k=1}^d (A u_j u_i^\dagger)_{kk} = \sum_{k=1}^d \left(\sum_{\ell=1}^d \langle u_k, A u_\ell \rangle \langle u_\ell, u_j u_i^\dagger u_k \rangle \right) \\ &= \sum_{k=1}^d \left(\sum_{\ell=1}^d \langle u_k, A u_\ell \rangle \delta_{ik} \delta_{\ell j} \right) \\ &= \langle u_i, A u_j \rangle = \langle u_i | A | u_j \rangle. \end{aligned}$$

(c) Now, given $f: \text{Her}(\mathbb{C}^d) \rightarrow \mathbb{R}$ and an ONB $(u_1, \dots, u_d)$ we construct A via

$$\langle u_i, A u_j \rangle = f(u_j u_i^\dagger).$$

Assume that f is linear and let $X \in \text{Her}(\mathbb{C}^d)$. Then:

$$\begin{aligned} \text{Tr}(AX) &= \sum_{k, \ell=1}^d \langle u_k, A u_\ell \rangle X_{\ell k} = \sum_{k, \ell=1}^d f(u_\ell u_k^\dagger) X_{\ell k} \\ &\stackrel{f \text{ is linear}}{=} \sum_{k, \ell=1}^d f(X_{\ell k} u_\ell u_k^\dagger) = f\left(\sum_{k, \ell=1}^d X_{\ell k} u_\ell u_k^\dagger\right) = f(X). \end{aligned}$$



Problem 2.2

Consider the following SDP:

$$\begin{aligned} & \text{minimise} && \text{Tr}(AX) \\ & \text{subject to} && \Phi_i(X) = B_i \quad ; i = 1, \dots, m \\ & && \Gamma_j(X) \preceq C_j \quad ; j = 1, \dots, m. \end{aligned}$$

Let α' be its optimal value.

Let X^* be an optimal solution of the original SDP, i.e.

$$\alpha' = \text{Tr}(AX^*) \leq \text{Tr}(AX) \quad \forall X \in F'.$$

Then, it is immediate that

$$\alpha = -\text{Tr}(AX^*) \geq -\text{Tr}(AX) \quad \forall X \in F'$$

and $\alpha' = -\alpha$.

Thus, X^* is also an optimal solution of

$$\begin{aligned} & \text{maximise} && -\text{Tr}(AX) \\ & \text{subject to} && \Phi_i(X) = B_i \quad i = 1, \dots, m \\ & && \Gamma_j(X) \preceq C_j \quad j = 1, \dots, m \end{aligned}$$

and the optimal value α' is the optimal value α up to a minus sign.

Problem 2.3

Consider the following problem:

$$\begin{aligned} &\text{maximise } \text{Tr}(H\rho) \\ &\text{subject to } \text{Tr}(\rho) = 1 \\ &\rho \geq 0 \end{aligned}$$

By the spectral theorem, we can decompose H as

$$H = \sum_{E \in \sigma(H)} E \Pi_E,$$

where

$$\begin{cases} \Pi_E \Pi_{E'} = \delta_{EE'} \Pi_E & \forall E, E' \in \sigma(H) \\ \sum_{E \in \sigma(H)} \Pi_E = \Pi_d \end{cases}$$

Denote by E^* the largest eigenvalue of H , i.e.

$$\forall E \in \sigma(H): E \leq E^*.$$

Then:

$$\begin{aligned} T_{\Pi}(H\rho) &= \sum_{E \in \sigma(H)} E T_{\Pi}(\Pi_E \rho) \leq E^* \sum_{E \in \sigma(H)} T_{\Pi}(\Pi_E \rho) = E^* T_{\Pi}[(\sum_E \Pi_E) \rho] \\ &= E^* T_{\Pi} \rho = E^* . \end{aligned}$$

From this, we conclude that

$$\forall \rho \in D(\mathbb{C}^d) : \quad \text{Tr}(\Pi(\rho)) \leq E^*$$

and hence

$$\alpha \leq E^*$$

Pick $\rho^* = \pi_{E^*} / \tau_\pi(\pi_{E^*})$. Then

$$T_{\Pi}(H\rho^*) = \sum_{E \in \mathcal{O}(H)} E \frac{T_{\Pi}(\overbrace{\Pi_E \Pi_{E^*}}^{\delta_{EE^*} \Pi_{E^*}})}{T_{\Pi}(\Pi_{E^*})} = E^* \frac{T_{\Pi}(\Pi_{E^*})}{T_{\Pi}(\Pi_{E^*})} = E^*.$$

Thus the optimal value α is the largest eigenvalue of the hamiltonian H .

Problem 2.4

Let H be hermitian with spectral decomposition

$$H = \sum_{E \in \sigma(H)} E \Pi_E.$$

Denote its smallest eigenvalue by $E_{\min}$. Then, for any density operator $\rho \in D(\mathbb{C}^d)$:

$$\text{Tr}(H\rho) = \sum_{E \in \sigma(H)} E \text{Tr}(\Pi_E \rho) \geq E_{\min} \text{Tr}\left[\left(\sum_{E \in \sigma(H)} \Pi_E\right) \rho\right] = E_{\min}.$$

This shows that $E_{\min}$ is a lower bound for the optimal value

$$\begin{aligned} \alpha &= \text{minimise } \text{Tr}(H\rho) \\ &\text{subject to } \text{Tr}(\rho) = 1 \\ &\rho \geq 0. \end{aligned}$$

Direct computation shows that $\rho^* = \Pi_{E_{\min}} / \text{Tr}(\Pi_{E_{\min}})$ satisfies $\text{Tr}(H\rho^*) = E_{\min}$. Hence, $\alpha = E_{\min}$ and ρ^* is an optimal solution.

Problem 2.5

Given a Hermitian operator $H \in \text{Her}(\mathbb{C}^d)$ and an integer $k \in \{1, \dots, d\}$, we must compute the sum of its k largest eigenvalues.

For simplicity, let us consider first the case in which $|\sigma(H)| = d$, i.e. all eigenspaces are one-dimensional. If $\{\pi_i\}_{i=1}^d$ are the projectors in the eigendecomposition of H , let $I_k \subseteq \{1, \dots, d\}$ be any collection of indices with k elements. Then

$$\sum_{i \in I_k} \text{Tr}(H \pi_i) = \sum_{i \in I_k} E_i \text{Tr}(\pi_i).$$

Since we've assumed one-dimensional eigenspaces, the RHS in the above eq. is the sum of the eigenvalues E_i , $i \in I_k$. Define the operator $\Pi_{I_k} = \sum_{i \in I_k} \pi_i$. Then, we showed that

$$\text{Tr}(H \Pi_{I_k}) = \sum_{i \in I_k} E_i$$

with

$$\text{Tr}(\Pi_{I_k}) = k$$

$$0 \leq \Pi_{I_k} \leq \Pi_d$$

If, by chance, we had chosen I_k corresponding to the largest eigenvalues, then we would be done.

So, I guess the SDP for our problem is

$$\begin{aligned} & \text{maximise } \text{Tr}(HX) \\ & \text{subject to } \text{Tr}(X) = k \\ & \quad 0 \leq X \leq I_d \end{aligned}$$

A formal proof that this is the correct SDP relies on the following:

Generalized
Purk's ineq. Lemma. Let $A, B \in \text{Her}(\mathbb{C}^d)$ and denote their eigenvalues by $\lambda_i(A)$ and $\lambda_i(B)$, respectively. Assume that they are ordered such that $\lambda_i(A) \geq \lambda_{i+1}(A)$ and $\lambda_i(B) \geq \lambda_{i+1}(B)$. Then:

$$\text{Tr}(AB) \leq \sum_{i=1}^d \lambda_i(A) \lambda_i(B).$$

If we set $A=H$ and $B=X$, then for any hermitian operator X :

$$\text{Tr}(HX) \leq \sum_{i=1}^d \lambda_i(H) \lambda_i(X).$$

If the spectrum of X satisfies $\sigma(X) \subseteq [0, 1]$, i.e. if

$$0 \leq X \leq \mathbb{I}_d,$$

then

$$\text{Tr}(HX) \leq \sum_{i=1}^{m_X} \lambda_i(H) \lambda_i(X)$$

that is, $\text{Tr}(HX)$ is upper bounded by the weighted sum of the first m largest eigenvalues of H where $m_X = |\text{supp}(X)|$ is the number of non-zero eigenvalues of X (with multiplicity).

Notice that if $\text{Tr}(X) = k$, then at least k $\lambda_i(X)$'s are non-zero i.e. $k \leq m_X$. Hence

$$\sum_{i=1}^{m_X} \lambda_i(H) \lambda_i(X) = \sum_{i=1}^k \lambda_i(H) \lambda_i(X) + \sum_{i=k+1}^{m_X} \lambda_i(H) \lambda_i(X).$$

The LHS is always equal or less than $\sum_{i=1}^k \lambda_i(H)$. To see that, consider their difference

$$\sum_{i=1}^k \lambda_i(H) - \sum_{i=1}^{m_X} \lambda_i(H) \lambda_i(X)$$

$$= \sum_{i=1}^k \lambda_i(H)(1 - \lambda_i(X)) - \sum_{i=k+1}^{m_X} \lambda_i(H) \lambda_i(X)$$

$$\geq \lambda_d(H) \left[\sum_{i=1}^k (1 - \lambda_i(X)) - \sum_{i=k+1}^{m_X} \lambda_i(X) \right] = \lambda_d(H) \left[k - \underbrace{\sum_{i=1}^{m_X} \lambda_i(X)}_{k - T\pi(X) = 0} \right] = 0.$$

$$\lambda_i(H) \geq \lambda_d(H)$$

Then,

$$\sum_{i=1}^k \lambda_i(H) \geq \sum_{i=1}^{m_X} \lambda_i(H) \lambda_i(X).$$

This implies

$$T\pi(HX) \leq \sum_{i=1}^k \lambda_i(H)$$

$$\forall X \text{ s.t. } X \in [0, \Pi_d] \text{ and } T\pi(X) = k. \quad \square$$

From this, it is immediate that the opt. value

$$\alpha = \text{maximise } T\pi(HX)$$

$$\text{subject to } T\pi(X) = k$$

$$0 \leq X \leq \Pi_d$$

is a lower bound for $\sum_{i=1}^k \lambda_i(H)$. A feasible X that saturates it is $X = \sum_{i=1}^k v_i v_i^t$, where $H = \sum_{i=1}^d \lambda_i(H) v_i v_i^t$.

Another way: apply the spectral thm to X .

Then:

$$\mathrm{Tr}(HX) = \sum_{i=1}^d \lambda_i(X) \langle x_i | H | x_i \rangle.$$

Let $\sigma(X) \subseteq [0, 1]$ and write $h_i \equiv \langle x_i | H | x_i \rangle$. Let $h_{[1]}, \dots, h_{[d]}$ be a non-increasing ordering of $h_1, \dots, h_d$. Then

$$\mathrm{Tr}(HX) \leq \sum_{i=1}^k h_{[i]} \leq \sum_{i=1}^k \lambda_i(H), \quad \blacksquare$$

$$\begin{aligned} \mathrm{Tr}(X) &= k \\ \sigma(X) &\subseteq [0, 1] \end{aligned}$$

$\hookrightarrow$ Schur-Horn Theorem

Problem 2.6

Recast

$$\text{maximise } \text{Tr}(AX) = \|A\|_1$$

$$\text{subject to } \|X\|_\infty \leq 1$$

as an SDP by exploiting that

$$\|T\|_\infty = \text{minimise } x$$

$$\text{subject to } -xI \leq T \leq xI$$

is an SDP.

Suppose X has operator norm $\|X\|_\infty \leq 1$. This means that

$$-\|X\|_\infty I_d \leq X \leq \|X\|_\infty I_d.$$

Since $\|X\|_\infty \leq 1$, then $-I_d \leq \|X\|_\infty I_d$ and $\|X\|_\infty I_d \leq I_d$. Thus $-I_d \leq X \leq I_d$.

Conversely, let $X \in [-I_d, I_d]$. The operator norm of X is such that if $-xI_d \leq X \leq xI_d$, then $\|X\|_\infty \leq x$.

Since we assumed $-I_d \leq X \leq I_d$, take $x=1$. Then $\|X\|_\infty \leq 1$.

Conclusion: $\|X\|_\infty \leq 1 \Leftrightarrow -I_d \leq X \leq I_d$. Thus:

$$\|A\|_1 = \text{maximise } \text{Tr}(AX)$$

$$\text{subject to } -I_d \leq X \leq I_d$$

Problem 2.7

Recall that

$$\|A\|_1 = \text{minimise } \text{Tr}(X) \\ \text{subject to } -X \leq A \leq X.$$

Let A^+ and A^- be the positive and negative parts of A and consider the following operator:

$$X^* = A^+ - A^-.$$

We have:

1) $A + X^* = (A^+ + A^-) + A^+ - A^- = 2A^+$. The latter is a positive op. because $A^+ \geq 0$. Thus $A + X^* \geq 0$, i.e.

$$-X^* \leq A.$$

2) $X^* - A = A^+ - A^- - (A^+ + A^-) = -2A^-$. Since $A^- \leq 0$, the latter is a positive operator because it is the product of A^- and a negative real number. Thus $X^* - A \geq 0$, i.e.

$$A \leq X^*.$$

From 1 and 2, it follows that X^* is feasible.

Also, notice that

$$\begin{aligned} \text{Tr}(X^*) &= \text{Tr}(A^+) + \text{Tr}(-A^-) \\ &= \sum_{\substack{\lambda(A) \in \sigma(A) \\ \lambda(A) \geq 0}} \lambda(A) + \sum_{\substack{\lambda(A) \in \sigma(A) \\ \lambda(A) < 0}} -\lambda(A) \\ &= \sum_{\lambda(A) \in \sigma(A)} |\lambda(A)| = \|A\|_1. \end{aligned}$$

Thus X^* is optimal.

Problem 2.8

We showed that

$$\begin{aligned} \|A\|_1 &= \text{maximise } \text{Tr}(AX) \\ \text{subject to } -I_d &\leq X \leq I_d. \end{aligned}$$

Write

$$A = \sum_{\substack{\lambda \in \sigma(A) \\ \lambda \geq 0}} \lambda \pi_\lambda + \sum_{\substack{\lambda \in \sigma(A) \\ \lambda < 0}} \lambda \pi_\lambda = A^+ + A^-$$

and define

$$\Pi^+ = \sum_{\substack{\lambda \geq 0 \\ \lambda \in \sigma(A)}} \pi_\lambda, \quad \Pi^- = \sum_{\substack{\lambda < 0 \\ \lambda \in \sigma(A)}} \pi_\lambda$$

and

$$X^* = \Pi^+ - \Pi^-.$$

We have:

$$1) \quad X^* + \Pi_d = (\Pi^+ - \Pi^-) + (\Pi^+ + \Pi^-) = 2\Pi^+ \geq 0. \text{ Thus } -\Pi_d \leq X^*.$$

$$2) \quad \Pi_d - X^* = (\Pi^+ + \Pi^-) - (\Pi^+ - \Pi^-) = 2\Pi^- \geq 0. \text{ Thus } X^* \leq \Pi_d.$$

From 1) and 2) we conclude that X^* is feasible.

Lastly, it is also optimal:

$$\begin{aligned} \text{Tr}(AX^*) &= \text{Tr}(A^+X^*) + \text{Tr}(A^-X^*) \\ &\stackrel{(1)}{=} \text{Tr}(A^+\Pi^+) - \text{Tr}(A^-\Pi^-) \\ &\stackrel{(2)}{=} \text{Tr}(\Pi^+A^+\Pi^+) - \text{Tr}(\Pi^-A^-\Pi^-) \\ &\stackrel{(3)}{=} \text{Tr}(A^+) - \text{Tr}(A^-) \\ &= \|A\|_1. \end{aligned}$$

In (1) we exploited the fact that $A^+\Pi^- = A^-\Pi^+ = 0$, and, in (2), the fact that $(\Pi^\pm)^2 = \Pi^\pm$, that the space is cyclic and that $[A^+, \Pi^+] = [A^-, \Pi^-] = 0$. 

Problem 2.9

Let $A, B \in \text{Pos}(\mathbb{C}^d)$. Apply the spectral theorem to B :

$$B = \sum_{b \in \sigma(B)} b \Pi_b.$$

Then

$$\text{Tr}(AB) = \sum_{b \in \sigma(B)} b \text{Tr}(A \Pi_b).$$

Recall that any ^{orthogonal} projection of rank π can be expressed as a sum of π rank one orth. projectors. Thus, if Π_b has rank π_b :

$$\Pi_b = \sum_{i=1}^{\pi_b} b_i b_i^\dagger$$

where $\{b_1, \dots, b_{\pi_b}\} \subseteq \mathbb{C}^d$ is a basis of the eigenspace associated with b . Hence

$$\begin{aligned} \text{Tr}(AB) &= \sum_{b \in \sigma(B)} b \text{Tr}\left[A \left(\sum_{i=1}^{\pi_b} b_i b_i^\dagger\right)\right] \\ &= \sum_{b \in \sigma(B)} \sum_{i=1}^{\pi_b} b \text{Tr}(A b_i b_i^\dagger) \\ &= \sum_{b \in \sigma(B)} \sum_{i=1}^{\pi_b} b \left[\sum_{k=1}^d \sum_{j=1}^d \langle e_k, A e_j \rangle \langle e_j, b_i \rangle \langle b_i, e_k \rangle \right] \\ &= \sum_{b \in \sigma(B)} \sum_{i=1}^{\pi_b} b \underbrace{\langle b_i, A b_i \rangle}_{\geq 0} \Rightarrow \boxed{\text{Tr}(AB) \geq 0} \end{aligned}$$

Problem 2.10

Dual SDP:

$$\begin{aligned} & \text{minimise} \quad \sum_{i=1}^m \text{Tr}(\gamma_i B_i) + \sum_{j=1}^m \text{Tr}(z_j C_j) \\ & \text{subject to} \quad \sum_i \Phi_i^\dagger(\gamma_i) + \sum_j \Gamma_j^\dagger(z_j) = A \\ & \quad \quad \quad z_j \geq 0 \quad \forall j \end{aligned}$$

Where $\gamma_i \in \text{Her}(\mathbb{C}^{d_i})$, $z_j \in \text{Her}(\mathbb{C}^{n_j})$, $\Phi_i: \text{Lim}(\mathbb{C}^d) \rightarrow \text{Lim}(\mathbb{C}^{d_i})$ and $\Gamma_j: \text{Lim}(\mathbb{C}^d) \rightarrow \text{Lim}(\mathbb{C}^{n_j})$.

Define the following variable $\tilde{\gamma} \in \text{Lim}(\bigoplus_{i=1}^m \mathbb{C}^{d_i} \oplus \bigoplus_{j=1}^m \mathbb{C}^{n_j})$:

$$\tilde{\gamma} = \gamma_1 \oplus \dots \oplus \gamma_m \oplus z_1 \oplus \dots \oplus z_m.$$

Then, the objective function becomes

$$\text{Tr} \left[\left(\bigoplus_{i=1}^m B_i \oplus \bigoplus_{j=1}^m C_j \right) \tilde{\gamma} \right].$$

Define $\tilde{\Phi}: \text{Lim}(\mathbb{C}^d) \rightarrow \text{Lim}(\bigoplus_{i=1}^m \mathbb{C}^{d_i} \oplus \bigoplus_{j=1}^m \mathbb{C}^{n_j})$ via

$$\tilde{\Phi}(x) = \Phi_1(x) \oplus \dots \oplus \Phi_m(x) \oplus \Gamma_1(x) \oplus \dots \oplus \Gamma_m(x).$$

If we represent an operator $T \in \text{Lim}(\bigoplus_{i=1}^m \mathbb{C}^{d_i} \oplus \bigoplus_{j=1}^m \mathbb{C}^{n_j})$ as

$$T = \begin{pmatrix} T_{d_1} & T_{d_1 d_2} & \dots & T_{d_1 d_m} & T_{d_1 n_1} & \dots & T_{d_1 n_m} \\ T_{d_2 d_1} & T_{d_2} & & & & & T_{d_2 n_m} \\ \vdots & & \ddots & & & & \vdots \\ & & & T_{d_m} & & & \\ & & & & T_{n_1} & & \\ & & & & & \ddots & \\ T_{n_m d_1} & T_{n_m d_2} & \dots & T_{n_m n_1} & \dots & & T_{n_m} \end{pmatrix}$$

where $T_{d_i} \in \text{Lim}(\mathbb{C}^{d_i})$, $T_{n_j} \in \text{Lim}(\mathbb{C}^{n_j})$, $T_{d_i n_j} \in \text{Lim}(\mathbb{C}^{n_j}; \mathbb{C}^{d_i})$, $T_{n_j d_i} \in \text{Lim}(\mathbb{C}^{d_i}; \mathbb{C}^{n_j})$,
 then

$$\tilde{\Phi}^{\dagger}(T) = \Phi_1^{\dagger}(T_{d_1}) + \dots + \Phi_m^{\dagger}(T_{d_m}) + \Gamma_1^{\dagger}(T_{n_1}) + \dots + \Gamma_n^{\dagger}(T_{n_m}).$$

Hence, the equality constraint becomes

$$\tilde{\Phi}(\tilde{Y}) = A.$$

Lastly, let $\tilde{\Gamma}_j : \text{Lim}(\bigoplus_{i=1}^m \mathbb{C}^{d_i} \oplus \bigoplus_{j=1}^n \mathbb{C}^{n_j}) \rightarrow \text{Lim}(\mathbb{C}^{n_j})$ be
 $\tilde{\Gamma}_j(T) \doteq T_{n_j}$. Then the inequality constraints become

$$\tilde{\Gamma}_j(\tilde{Y}) \succeq 0.$$

The compact dual SDP is

$$\begin{aligned} & \text{minimise } T_n(\tilde{D} \tilde{Y}) \\ & \text{subject to } \tilde{\Phi}(\tilde{Y}) = A \\ & \quad \tilde{\Gamma}_j(\tilde{Y}) \succeq 0 \quad \forall j. \end{aligned}$$

Problem 2.11

Consider the following SDP

$$\begin{aligned} & \text{minimise } \text{Tr}(AX) \\ & \text{subject to } \Phi_i(X) = B_i, \quad i = 1, \dots, m \\ & \quad \Gamma_j(X) \geq C_j, \quad j = 1, \dots, n. \end{aligned}$$

(a) To construct its dual, consider the following Lagrangian:

$$\begin{aligned} L(X, \vec{Y}, \vec{Z}) = & \text{Tr}(AX) + \sum_{i=1}^m \text{Tr}[Y_i(B_i - \Phi_i(X))] \\ & - \sum_{j=1}^n \text{Tr}[Z_j(\Gamma_j(X) - C_j)] \end{aligned}$$

(b) Suppose that X is feasible. Then

$$L(X, \vec{Y}, \vec{Z}) = \text{Tr}(AX) - \sum_{j=1}^n \text{Tr}(Z_j(\Gamma_j(X) - C_j)).$$

Now, if $Z_j \geq 0$

$$\text{Tr}(Z_j(\Gamma_j(X) - C_j)) \geq 0$$

because $\Gamma_j(X) - C_j$ is positive semidefinite

for all z . Hence

$$z_j \geq 0 \Rightarrow L(x, \vec{y}, \vec{z}) = \text{Tr}(AX) \\ \forall \text{ feasible } x.$$

(c) We can rewrite L as:

$$L = \text{Tr}\left[\left(A + \sum_{i=1}^m \Phi_i^T(y_i) - \sum_{j=1}^m \Gamma_j^T(z_j)\right)x\right] + \sum_{i=1}^m \text{Tr}(y_i B_i) - \sum_{j=1}^m \text{Tr}(z_j C_j).$$

To make independent of x , we impose

$$\sum_{i=1}^m \Phi_i^T(y_i) - \sum_{j=1}^m \Gamma_j^T(z_j) = A$$

(d) Dual SDP:

$$\begin{aligned} &\text{maximise} \quad \sum_{i=1}^m \text{Tr}(y_i B_i) - \sum_{j=1}^m \text{Tr}(z_j C_j) \\ &\text{subject to} \quad \sum_{i=1}^m \Phi_i^T(y_i) - \sum_{j=1}^m \Gamma_j^T(z_j) = A \\ &\quad \quad \quad z_j \geq 0 \end{aligned}$$

Problems 2.12 to 2.14

① Largest eigenvalue SDP:

$$\begin{aligned} \text{Primal:} \quad & \text{maximise } \text{Tr}(H\rho) \\ & \text{subject to } \text{Tr}(\rho) = 1 \\ & \rho \geq 0 \end{aligned}$$

Our dual variables will be $\gamma \in \mathbb{R}$ and $Z \in \text{Her}(\mathbb{C}^d)$.

The adjoint map of $\text{Tr}: \text{Lin}(\mathbb{C}^d) \rightarrow \text{Lin}(\mathbb{C}^1)$ satisfies by definition

$$\begin{aligned} \langle \text{Tr}^\dagger(a), B \rangle_{\text{HS}} &= \langle a, \text{Tr}(B) \rangle_{\mathbb{C}} = \bar{a} \text{Tr}(B) \\ &= \text{Tr}([\text{Tr}^\dagger(a)]^\dagger B) \end{aligned}$$

The adjoint of the identity channel $\text{id}: \text{Lin}(\mathbb{C}^d) \rightarrow \text{Lin}(\mathbb{C}^d)$ is itself. Bearing these in mind, we can write the dual SDP as:

$$\begin{aligned} & \text{minimise } \gamma - \text{Tr}(Z H) \\ & \text{subject to } \text{Tr}^\dagger(\gamma) - Z = H \\ & Z \geq 0. \end{aligned}$$

Now, notice that $a \in \mathbb{C} \mapsto a \mathbb{I}_d$ satisfies the adjoint condition. By uniqueness of the adjoint map, $\text{Tr}^\dagger(a) = a \mathbb{I}_d$.

Also, $\text{Tr}(Z0) = 0$ for any Z . Thus the dual SDP is

$$\begin{aligned} &\text{minimise } \gamma \\ &\text{subject to } \gamma \mathbb{I}_d - Z = H \\ &\quad Z \geq 0. \end{aligned}$$

Since Z is not part of the (dual) objective function, we can simplify even further to

Dual:

$$\begin{aligned} &\text{minimise } \gamma \\ &\text{subject to } \gamma \mathbb{I}_d \geq H \end{aligned}$$

② Sum of largest k eigenvalues:

Primal: maximise $\text{Tr}(HX)$

subject to $0 \leq X \leq \mathbb{I}$

$$\text{Tr}(X) = k$$

Dual variables: γ, Z_0, Z_1 . Maps: $\text{Tr}: \text{Lin}(\mathbb{C}^d) \rightarrow \mathbb{C}$,

$-\text{id}, \text{id}: \text{Lin}(\mathbb{C}^d) \rightarrow \text{Lin}(\mathbb{C}^d)$. Then, the dual is

minimise $k\gamma + \text{Tr}(Z_1)$

subject to $\gamma \mathbb{I} - Z_0 + Z_1 = H$

$$Z_0 \geq 0$$

$$Z_1 \geq 0$$

z_0 is again a slack variable, thus the dual can be simplified to

Dual:
$$\begin{aligned} & \text{minimise } \lambda \gamma + \text{Tr}(z) \\ & \text{subject to } \gamma \Pi + z \geq H \\ & \quad z \geq 0 \end{aligned}$$

③ Relaxed Hamiltonian feasibility problem

Primal:
$$\begin{aligned} & \text{maximise } \lambda \\ & \text{subject to } \text{Tr}(H \rho_i) = E_i, i=1, \dots, m \\ & \quad H - \lambda \Pi \geq 0 \end{aligned}$$

It is better to define $x = H \oplus \lambda$.

- $\lambda = \text{Tr}[(0_d \oplus 1)x]$
- $\text{Tr}(H \rho_i) = \text{Tr}[(\rho_i \oplus 0)x] = \Phi_i(x)$
- $-H + \lambda \Pi = \Gamma(x)$

We have $\Phi_i: \text{Lin}(\mathbb{C}^d \oplus \mathbb{C}) \rightarrow \mathbb{C}$ and $\Gamma: \text{Lin}(\mathbb{C}^d \oplus \mathbb{C}) \rightarrow \text{Lin}(\mathbb{C}^d)$.

• Adjoints:

$\hookrightarrow \text{Tr}[(\Phi_i^\dagger(x))^\dagger P] = \bar{x} \text{Tr}[(\rho_i \oplus 0)P] \Rightarrow \Phi_i^\dagger(x) = (x \rho_i) \oplus 0$

$\hookrightarrow$ Write $P \in \text{Lin}(\mathbb{C}^d \oplus \mathbb{C})$ as $P = \begin{pmatrix} A & v \\ v^\dagger & \alpha \end{pmatrix}$ where $A \in \text{Lin}(\mathbb{C}^d)$, $\alpha \in \mathbb{C}$, $v, v^\dagger \in \mathbb{C}^d$.

Then $\Gamma(P) = -A + \alpha \Pi$. The adjoint acts as

$$\begin{aligned} \text{Tr}[(\Gamma^T(Q))^T P] &\doteq \text{Tr}[Q^T \Gamma(P)] = \text{Tr}[Q^T (-A + \alpha I)] \\ &= -\text{Tr}(Q^T A) + \alpha \text{Tr}(Q^T) \\ &= \text{Tr}\left[\begin{pmatrix} -Q^T & 0 \\ 0 & \text{Tr}(Q^T) \end{pmatrix} P\right] \end{aligned}$$

$$C_0 = \text{Tr}\left[\begin{pmatrix} -Q & 0 \\ 0 & \text{Tr}(Q) \end{pmatrix} P\right]$$

$$\text{Thus: } \Gamma^T(Q) = -Q \oplus \text{Tr}(Q).$$

$$\begin{aligned} \text{Dual: } & \text{minimise } \sum_{i=1}^m E_i \gamma_i \\ & \text{subject to } \sum_{i=1}^m \gamma_i p_i \oplus 0 + (-Z \oplus \text{Tr}(Z)) = 0_6 \oplus 1 \\ & Z \geq 0 \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \text{minimise } \sum_{i=1}^m E_i \gamma_i \\ & \text{subjected to } \sum_{i=1}^m \gamma_i \text{Tr}(p_i) = 1 \\ & \sum_{i=1}^m \gamma_i p_i \geq 0 \end{aligned}$$

④ Minimum eigenvalue problem

$$\begin{aligned} \text{Primal: } & \text{maximise } -\text{Tr}(A X) \\ & \text{subject to } \text{Tr}(X) = 1 \\ & X \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Lagrangian: } L &= -\text{Tr}(A X) + \gamma(1 - \text{Tr}(X)) + \text{Tr}(Z X) \\ &= \text{Tr}[-(A + Z - \gamma I) X] + \gamma \end{aligned}$$

Dual:

$$\begin{array}{ll} \text{minimise} & \gamma \\ \text{subject to} & \gamma \mathbb{I} \geq -A. \end{array}$$

Problem 2.15

$$\begin{array}{ll} \text{Primal:} & \text{maximise } \text{Tr}(H\rho) \\ & \text{subject to } \text{Tr}(\rho) = 1 \\ & \rho \geq 0 \end{array}$$

$$\begin{array}{ll} \text{Dual:} & \text{minimise } \gamma \\ & \text{subject to } \gamma \mathbb{I}_d \geq H \end{array}$$

Let $\varepsilon > 0$ and denote by λ_H^* the largest eigenvalue of H . Then, we have

$$(\lambda_H^* + \varepsilon) \mathbb{I} \geq H,$$

i.e. strict feasibility holds for the dual.

For the primal SDP, just take $\rho = \frac{1}{d} \mathbb{I}_d$. This operator is (strictly) positive with all eigenvalues equal to $1/d$, and it has unit trace. Hence, the primal is strictly feasible too. Conclusion: strong duality holds.

Problem 2.16

Recall that

$$\|A\|_\infty = \text{minimise } x$$

$$\text{subject to } -xI \leq A \leq xI.$$

(a) Let's find its dual: variables z_1 and z_2 , $z_i \geq 0$.

Channels:

$$\Gamma_1: \mathbb{C} \rightarrow \text{Lim}(\mathbb{C}^d) \quad x \mapsto xI_d,$$

$$\Gamma_2: \mathbb{C} \rightarrow \text{Lim}(\mathbb{C}^d) \quad x \mapsto xI_d.$$

The duals are $\Gamma_i^*(x) = \text{Tr}(x)$ and the Lagrangian is

$$L = x - \text{Tr}[z_1(\Gamma_1(x) + A)] - \text{Tr}[z_2(\Gamma_2(x) - A)].$$

From by the usual procedure, we obtain the dual SDP:

$$\text{maximise } \text{Tr}[(z_2 - z_1)A]$$

$$\text{subject to } \text{Tr}(z_1 + z_2) = 1 \quad (1)$$

$$z_1 \geq 0, z_2 \geq 0.$$

(b) The operator norm is also given by

$$\begin{aligned} \|A\|_\infty &\doteq \text{maximise } \text{Tr}(AX) \\ &\text{subject to } \|X\|_1 \leq 1. \end{aligned} \quad (2)$$

Comparing (1) and (2), we suspect that

$$\{X \mid \|X\|_1 \leq 1\} = \{X \mid X = Z_2 - Z_1, \text{Tr}(Z_1 + Z_2) = 1, Z_1 \geq 0, Z_2 \geq 0\}.$$

Well, if X belongs to the set in the RHS, then

$$\|X\|_1 = \|Z_2 - Z_1\|_1 \leq \|Z_2\|_1 + \|Z_1\|_1 = \text{Tr}(Z_2) + \text{Tr}(Z_1) = 1.$$

Thus $\|X\|_1 \leq 1$.

Now, suppose that X satisfies $\|X\|_1 \leq 1$. Since

$$\begin{aligned} \|X\|_1 &= \text{minimise } \text{Tr}(Y) \\ &\quad -Y \leq X \leq Y \end{aligned}$$

$\exists Y^*$ s.t. $-Y^* \leq X \leq Y^*$, $\text{Tr}(Y^*) = \|X\|_1$. One possible choice is

$$Y^* = X^+ + X^-, \text{ where } X^+ = \Pi^+ X \Pi^+ \text{ and } X^- = -\Pi^- X \Pi^-,$$

$\Pi^\pm$ being the orth. projections onto non-negative and negative eigenspaces of X .

Notice that $X = X^+ - X^-$ and $\text{Tr}(Y^*) = \|X\|_1$.

Now, since $\|X\|_1 \leq 1$, the op. $\frac{(1-\|X\|_1)}{d} \mathbb{I}$ is PSD.
Thus, we can write

$$X = \underbrace{\left[X^+ + \left(\frac{1-\|X\|_1}{2d} \right) \mathbb{I} \right]}_{Z_2} - \underbrace{\left[\left(\frac{1-\|X\|_1}{2d} \right) \mathbb{I} + X^- \right]}_{Z_1}.$$

Clearly Z_i is positive, since it is the sum of positive operators. Also:

$$\text{Tr}(Z_1 + Z_2) = \text{Tr}(Y^*) + 1 - \|X\|_1 = \|X\|_1 + 1 - \|X\|_1 = 1.$$

Hence, X admits the desired decomposition.

So:

$$\begin{aligned} \|A\|_\infty &= \text{maximise } \text{Tr}[A(Z_2 - Z_1)] \\ &\text{subject to } \text{Tr}(Z_1 + Z_2) = 1 \\ &Z_1 \geq 0, Z_2 \geq 0. \end{aligned}$$

(c) We already proved both inclusions in (b).

$$B_1 = \left\{ X \in \text{Her}(\mathbb{C}^d) \mid X = Z_2 - Z_1, \text{Tr}(Z_1 + Z_2) = 1, Z_i \geq 0, i=1,2 \right\}.$$

Problem 2.17

- ε -ball for operator norm:

$$B_\varepsilon^\infty = \{W \in \text{Her}(\mathbb{C}^d) \mid \|W\|_\infty < \varepsilon\}.$$

$$\|W\|_\infty < \varepsilon \Rightarrow \left\| \frac{1}{\varepsilon} W \right\|_\infty \leq 1$$

$$\Leftrightarrow \frac{1}{\varepsilon} W = Z_2 - Z_1, Z_2 + Z_1 = \mathbb{I}, Z_i \geq 0$$

$$\Leftrightarrow W = \tilde{Z}_2 - \tilde{Z}_1, \tilde{Z}_1 + \tilde{Z}_2 = \varepsilon \mathbb{I}, \tilde{Z}_i = \varepsilon Z_i \geq 0.$$

$$B_\varepsilon^\infty = \{W \mid W = Z_2 - Z_1, Z_2 + Z_1 = \varepsilon \mathbb{I}, Z_i \geq 0, i=1,2\}.$$

- ε -ball for trace norm:

$$W \in B_\varepsilon^1 \Leftrightarrow \left\| \frac{1}{\varepsilon} W \right\|_1 \leq 1$$

$$\Leftrightarrow \frac{1}{\varepsilon} W = Z_2 - Z_1, \text{Tr}(Z_1 + Z_2) = 1, Z_i \geq 0$$

$$\Leftrightarrow W = \tilde{Z}_2 - \tilde{Z}_1, \text{Tr}(\tilde{Z}_1 + \tilde{Z}_2) = \varepsilon, \tilde{Z}_i = \varepsilon Z_i \geq 0.$$

$$B_\varepsilon^1 = \{W \mid W = Z_2 - Z_1, \text{Tr}(Z_1 + Z_2) = \varepsilon, Z_i \geq 0, i=1,2\}.$$

Constraints like $\|X\|_1 \leq \epsilon$ or $\|X\|_\infty \leq \epsilon$ can be replaced by introducing Z_1, Z_2 and the appropriate linear constraints.

Problem 2.18

$$\begin{aligned} \text{SDP: } & \text{maximise } \text{Tr}(A(Z_2 - Z_1)) \\ & \text{s.t. } Z_1 + Z_2 = I \\ & Z_i \geq 0, i=1,2 \end{aligned}$$

Take $Z_1 = \Pi^-$ and $Z_2 = \Pi^+$. $A = \Pi^+ A \Pi^+ + \Pi^- A \Pi^-$ and $\Pi^+ \Pi^- = \Pi^- \Pi^+ = 0$. Hence:

$$\begin{aligned} \text{Tr}(A(\Pi^+ - \Pi^-)) &= \text{Tr}(\Pi^+ \overbrace{A \Pi^+}^{\Pi^+}) - \text{Tr}(\Pi^- \overbrace{A \Pi^-}^{\Pi^-}) \\ &= \text{Tr}(A^+) - \text{Tr}(A^-) \\ &= \sum_{\substack{\lambda_i \geq 0 \\ \lambda_i \in \sigma(A)}} \lambda_i + \sum_{\substack{\lambda_i < 0 \\ \lambda_i \in \sigma(A)}} (-\lambda_i) = \sum_{\lambda_i \in \sigma(A)} |\lambda_i| = \|A\|_1. \end{aligned}$$

(Π^+, Π^-) is clearly feasible: $\Pi^\pm \in \text{Pos}(\mathbb{C}^d)$ and $\Pi^+ + \Pi^- = I_d$.

Problem 2.19

$$\begin{aligned} \text{SDP: } & \text{maximise } \text{Tr}(A(Z_2 - Z_1)) \\ & \text{s.t. } \text{Tr}(Z_1 + Z_2) = 1 \\ & Z_i \geq 0, i=1,2 \end{aligned}$$

Suppose $\|A\|_{\infty} = \lambda$, $\lambda \geq 0$, $\lambda \in \sigma(A)$. Then, let v be an eigenvector such that $Av = \lambda v$, $\langle v, v \rangle^{\frac{1}{2}} = 1$. Choose $Z_1 = 0$ and $Z_2 = \langle v, \cdot \rangle v$ (orth. proj. onto v). Then

$$\langle u, Z_2 u \rangle = \langle u, \langle v, u \rangle v \rangle = |\langle u, v \rangle|^2 \geq 0 \quad \forall u.$$

Thus $Z_2 \geq 0$.

$$\text{Tr}(Z_1 + Z_2) = \text{Tr}(Z_2) = 0 + 1 = 1.$$

So, (Z_1, Z_2) is feasible.

Since $Av = \lambda v$:

$$\text{Tr}[A(Z_2 - Z_1)] = \text{Tr}[\langle v, \cdot \rangle Av] = \text{Tr}(\lambda Z_2) = \lambda.$$

If $\|A\|_{\infty} = -\lambda$, $\lambda < 0$, $\lambda \in \sigma(A)$, take $Z_2 = 0$ and $Z_1 = \langle v, \cdot \rangle v$, where $Av = \lambda v$, $\langle v, v \rangle^{\frac{1}{2}} = 1$. 

Problem 2.20

Primal: maximize $\text{Tr}(AX)$
 s.t. $-\mathbb{I} \leq X \leq \mathbb{I}$

Maps: $\Gamma_1 = -\text{id}$, $\Gamma_2 = \text{id}$.

(a) Dual: minimize $\text{Tr}(Z_1 + Z_2)$
 subject to $\Gamma_1^\dagger(Z_1) + \Gamma_2^\dagger(Z_2) = A$
 $Z_1 \geq 0, Z_2 \geq 0$

Since $\Gamma_1^\dagger = \Gamma_1$ and $\Gamma_2^\dagger = \Gamma_2$:

Dual: minimize $\text{Tr}(Z_1 + Z_2)$
 subject to $Z_2 - Z_1 = A$
 $Z_1 \geq 0, Z_2 \geq 0$ 

(b) For an arbitrary $A \in \text{Her}(\mathbb{C}^d)$, it belongs to a ball of radius ε if

$$\begin{aligned} A &= Z_2 - Z_1 \\ \text{Tr}(Z_2 + Z_1) &= \varepsilon \\ Z_i &\geq 0 \end{aligned}$$

So, the smallest ball containing A is B_{π^*} , where

$$\begin{aligned} \pi_* &= \text{minimise } \text{Tr}(Z_2 - Z_1) \\ \text{s.t. } & Z_2 + Z_1 = A \\ & Z_1 \geq 0, Z_2 \geq 0. \end{aligned}$$

Since the optimal value is $\|A\|_1$, the SDP tells that the smallest ball possible is $B_{\|A\|_1}$. In particular, B_1 is the smallest ball that contains operators A s.t. $\|A\|_1 \leq 1$.

Problem 2.21

$$\begin{aligned} P: & \text{minimise } \text{Tr}(X) \\ \text{s.t. } & -X \leq A \leq X \\ X^* &= A^+ - A^- \end{aligned}$$

$$\begin{aligned} D: & \text{maximise } \text{Tr}[A(Z_2 - Z_1)] \\ & Z_1 + Z_2 = \Pi \\ & Z_1 \geq 0, Z_2 \geq 0 \\ Z_1^* &= \Pi^-, Z_2^* = \Pi^+ \end{aligned}$$

Comp. Slackness conditions:

$$\begin{aligned} Z_1^*(X^* + A) &= 0 \quad (i) \\ Z_2^*(X^* - A) &= 0 \quad (ii) \end{aligned}$$

$$(i): \Pi^- (A^+ - A^- + A^+ + A^-) = 2\Pi^- A^+ = 2(\underbrace{\Pi^- \Pi^+}_0) A \Pi^+ = 0$$

$$(ii): \Pi^+ (A^+ - A^- - A^+ - A^-) = -2\Pi^+ A^- = -2(\underbrace{\Pi^+ \Pi^-}_0) A \Pi^- = 0.$$

Problem 2.22

$$(a) \quad X = \begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix} \Rightarrow \lambda \mathbb{I}_2 - X = \begin{pmatrix} \lambda - a & -c \\ -\bar{c} & \lambda - b \end{pmatrix}$$

$$\Rightarrow p_X(\lambda) = |\lambda \mathbb{I}_2 - X| = \lambda^2 - (a+b)\lambda + ab - |c|^2.$$

The eigenvalues are:

$$\begin{aligned} \lambda_{\pm} &= \frac{1}{2} \left(a+b \pm \sqrt{a^2 + 2ab + b^2 - 4ab + 4|c|^2} \right) \\ &= \frac{1}{2} \left(a+b \pm \sqrt{(a-b)^2 + 4|c|^2} \right) \end{aligned}$$

(b) Suppose $\text{Tr}(X) = a+b = 1$. For 2×2 matrices,

$X \geq 0$ iff $\text{Tr}(X) \geq 0$ and $\det X \geq 0$. Since $\det X = ab - |c|^2$ and $b = 1-a$:

$$X \geq 0 \Leftrightarrow a(1-a) \geq |c|^2.$$

